INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Second Year, 2022-23, Introduction to Linear Models
Mid-semesteral Examination, February 24, 2023
Marks are shown in square brackets. **Total Marks: 50**
Time: $2\frac{1}{2}$ Hours

1. Suppose (X, Y) is bivariate normal with $E(X) = 0 = E(Y)$, $Var(X) = \sigma^2 = Var(Y)$ and a correlation coefficient of $\rho = 0.4$ between X and Y. Define $U = (X - Y)/\sqrt{1-\rho}$ and $V = (X+Y)/\sqrt{1+\rho}$. Let P be a 2×2 symmetric, idempotent matrix. Find the probability distribution of $(U, V)\, P\, (U, V)'$. [10]

2. Suppose $Z \sim N(0, 1)$ independent of U which takes values 1 and -1 with equal probability. Let $Y = UZ$.
(a) Find the probability distribution of Y.
(b) Find the covariance between Y and Z.
(c) Are Y and Z independent? Justify. [2+5+3]

3. Consider the model $\mathbf{Y} = X\beta + \epsilon$, where $X_{n\times p}$ has rank p; also $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$. Let $\hat{\beta}$ be the least squares estimate of β. Consider any matrix $A_{q\times p}$ of rank q.
(a) Find the probability distribution of $(\hat{\beta} - \beta)' A' (A(X'X)^{-1}A')^{-1} A(\hat{\beta} - \beta)$.
(b) Find $E\left[\hat{\beta}' A' (A(X'X)^{-1}A')^{-1} A\hat{\beta}\right]$; how does it compare with $q\sigma^2$? [7+8]

4. Consider the following model:
$y_1 = \alpha + \phi + \gamma + \epsilon_1$
$y_2 = \alpha + \phi - \gamma + \epsilon_2$
$y_3 = 2\alpha + 2\phi + \gamma + \epsilon_3$
$y_4 = 2\alpha + 2\phi - \gamma + \epsilon_4$
where α, ϕ, γ are unknown regression parameters and ϵ_i are uncorrelated random variables having mean 0 and variance σ^2.
(a) Among the regression parameters which ones are estimable?
(b) What is the BLUE of the estimable regression parameters?
(c) What is the variance of the BLUEs above? [6+6+3]